Random Sums and Graphs

Probability

Events, probabilities,
conditional probabilities,
union bound

Random variables,
expected value,
linearity of expectation

Markov's inequality

Algorithms

Approx. 3-SAT,

Big idea: we only want the average

Linearity of Expectation

$E[X+Y] = E[X] + E[Y]$ (ridiculously useful)

"average sum = sum of averages"

let $Y_i = \begin{cases} 1 & \text{is } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$ $(i = 1, \dots, m)$

$E[Y_i] = 7/8$ (1 clause case)

$E\left[\begin{smallmatrix}\#\text{clauses} \\ \text{satisfied}\end{smallmatrix}\right] = E\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} E[Y_i] = 7/8\, m$

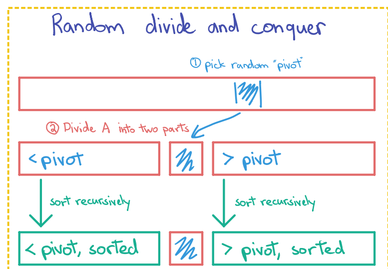
let $Y_i = \begin{cases} 1 & \text{is } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$

$Z = \sum_{i=1}^{m} Y_i = \#$ clauses satisfied

$E[Z] = \frac{1}{2} E[Z \mid x_1 = T] + \frac{1}{2} E[Z \mid x_1 = F]$

$\Rightarrow$ either $E[Z \mid x_1 = T]$ or $E[Z \mid x_1 = F]$ is $\geq E[Z]$!

Quicksort,



Expected running time of QuickSort

For each $i, j \in [n]$ w/ $i < j$,

$X_{ij} = \begin{cases} 1 & \text{if } i\text{th smallest element is compared to } j\text{th sma}\dots \\ 0 & \text{otherwise} \end{cases}$

$\sum_{i<j} X_{ij}$ = total # comparisons made by the algorithm

$\approx$ total running time

$E\left[\sum_{i<j} X_{ij}\right]$ = average running time

$E\left[\sum_{i<j} X_{ij}\right] = \sum_{i,j} E[X_{ij}]$

Linearity of Expectation

$E[X+Y] = E[X] + E[Y]$ (ridiculously useful)

"average sum = sum of averages"

now analyze $E[X_{ij}]$ in isolation

$E[X_{ij}] = \Pr[X_{ij} = 1]$

$= \Pr\left[\begin{smallmatrix}i\text{th element is} \\ \text{compared to } j\text{th}\end{smallmatrix}\right]$

i ... j

ith elem is compared w/ jth

$\Leftrightarrow$ ith or jth elem selected before any element in between

happens w/ probability $\frac{2}{j-i+1}$

$E[X_{ij}] = \frac{2}{j-i+1}$

$E\left[\sum_{i<j} X_{ij}\right] = \sum_{i,j} E[X_{ij}]$ by linearity of expectation

$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = \sum_{j=1}^{n} \sum_{i=1}^{j} \frac{2}{j-i+1}$

$\leq 2n\left(\frac{1}{1} + \frac{1}{2} + \dots \frac{1}{n}\right)$

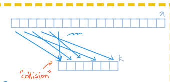
$\leq O(n \log n)$ nth harmonic number

Quickselect (HW)

Probability

Hash Functions,

Universal Hash Functions



$h: [n] \to k$ s.t. for all pairs $i, j \in [n]$,

$\Pr[h(i) = h(j)] \le \frac{1}{k}$ (much weaker than ideal hash)

e.g. $h(x) = ((ax + b) \bmod p) \bmod k$

where
- p prime, $> k$
- a in $\{1, \ldots, p\}$ unif. random
- b in $\{0, \ldots, p-1\}$ unif. random

Markov's Inequality

Markov's Inequality (special case)

Let $X \ge 0$ be nonnegative RV

$$\Pr[X \ge 2E[X]] \le \frac{1}{2}$$

Algorithms

Frequency Estimation, Heavy Hitters (in streams)

A_1, h_1

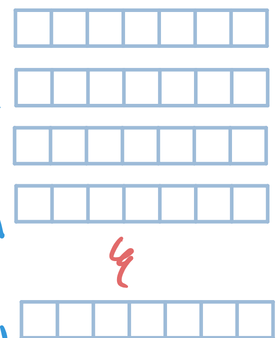
A_2, h_2

A_3, h_3

A_4, h_4

$\vdots$

A_d, h_d



estimate f_i w/ $\min_{j=1,\ldots,d} A_j[h_j(i)]$

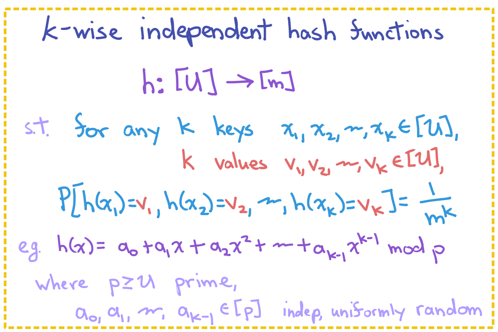
Probability

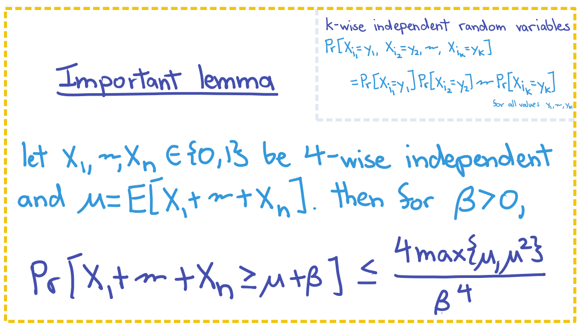
k-wise independence, higher order moments

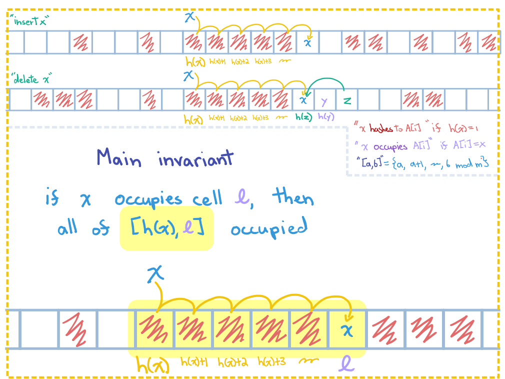
(generalizating universal hash functions and Markov's inequality)

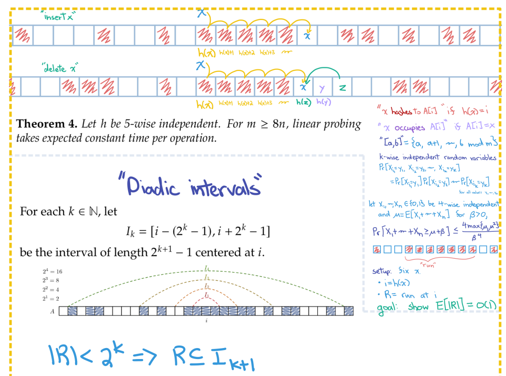
Algorithms

Hash tables w/ chaining + linear probing





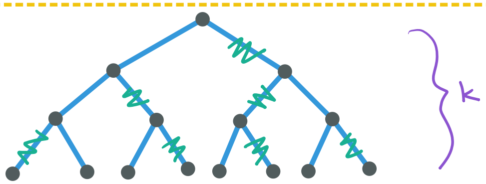




Probability

Algorithms

Galton-Watson processes

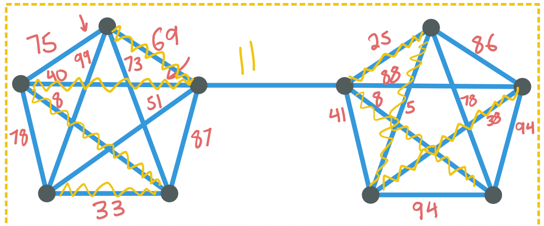


let T = complete binary tree of height k,
w/ every edge deleted w/ prob $\leq 1/2$

Fact: $P[\text{some leaf connected to root}] \geq \frac{1}{k}$

"Galton-Watson process"

Random Contractions for min cut



Randomized contractions

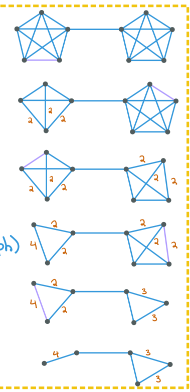
Input: $G=(V,E)$, weights $c \in \mathbb{R}^E_{\geq 0}$ (capacities)

① while $|E| > 1$

Ⓐ sample $e \in E$ in proportion to c

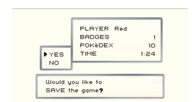
Ⓑ "contract" e

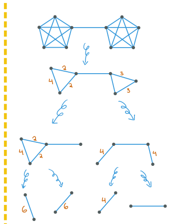
② return set of edges (in original graph) contracted to remaining edge



Amplification by branching

Instead of restarting at the beginning, restart partway through





High-level algorithm

① randomly contract until prob. of ruining the min-cut is $\sim 1/2$

② Branch:

make two copies of current graph

recurse on both copies

Probability	Algorithms (/ results)
Multiplicative Chernoff Bound	Random graphs & the giant component

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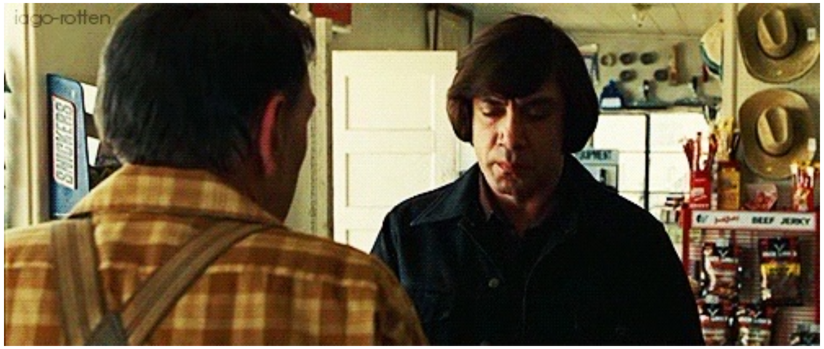
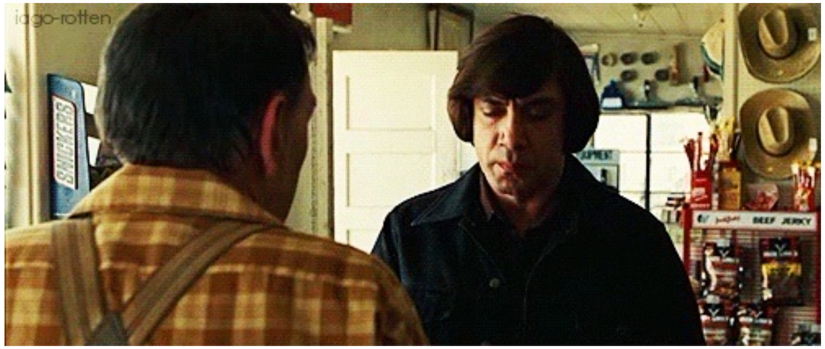


Flip 100 coins.

50 come up heads.

Should you be surprised?

(Maybe a little: ≈ 8% chance)

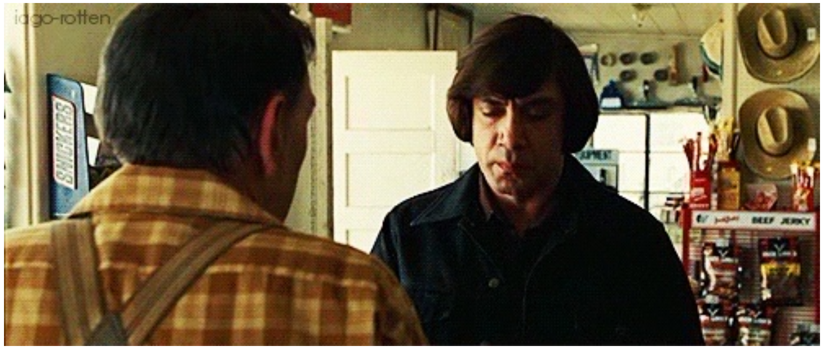


Flip 100 coins.

0 come up heads.

Should you be surprised?

Absolutely: $\frac{1}{2^{100}}$ chance

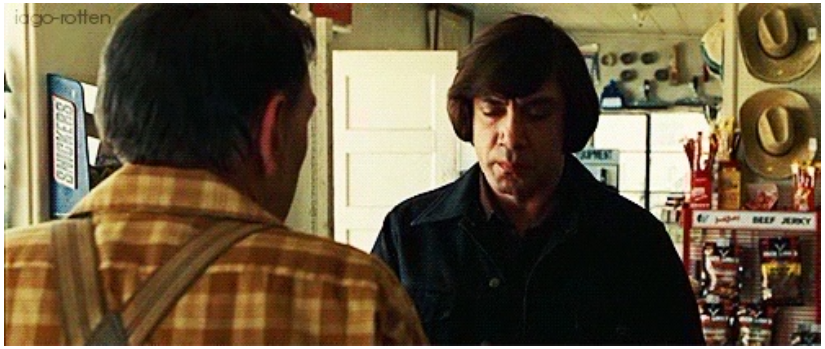


Flip 100 coins.

≤ 25 come up heads.

Should you be surprised?

Yes: $\approx 2.818 \times 10^{-7}$ %

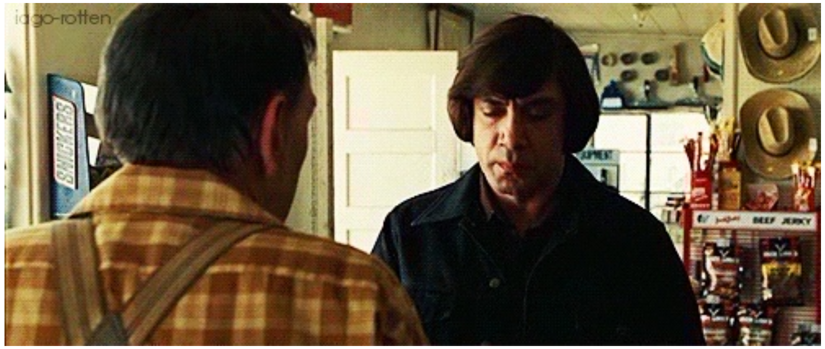


Flip 10 coins.

≤ 4 come up heads.

Should you be surprised?

No: $\approx 37.7\%$

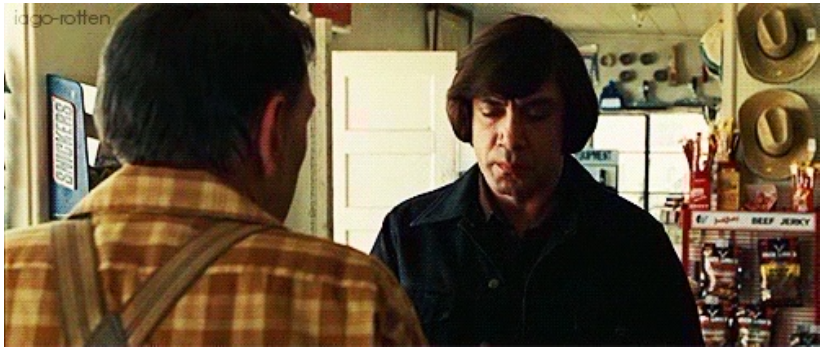


Flip 1000 coins.

$\leq 40\%$ come up heads.

Should you be surprised?

Yes: $\approx 1.364 \times 10^{-10}$ probability



10 coins.

≤ 4 heads.

$\approx 37.7\%$

1000 coins.

$\leq 40\%$ heads.

1.364×10^{-10} prob.

scale matters

Concentration phenomena

let $X_1, \dots, X_n \in [0,1]$ be independent

$$\mu = E[X_1 + \dots + X_n]$$

Q: probability that

$X_1 + \dots + X_n$ is close to μ?
(multiplicatively)

A: error probability exponentially small in μ

Multiplicative Chernoff bound

let $X_1, \dots, X_n \in [0,1]$ be independent

$\mu = E[X_1 + \dots + X_n]$

$\varepsilon \in (0,1)$

- $P[X_1 + \dots + X_n \geq (1+\varepsilon)\mu] \leq e^{-\varepsilon^2 \mu / 3}$
- $P[X_1 + \dots + X_n \leq (1-\varepsilon)\mu] \leq e^{-\varepsilon^2 \mu / 2}$

(proof at end if time permits)

Lecture 4 (last Thursday)

Probability	Algorithms (/ results)
Multiplicative Chernoff Bound	Random graphs & the giant component

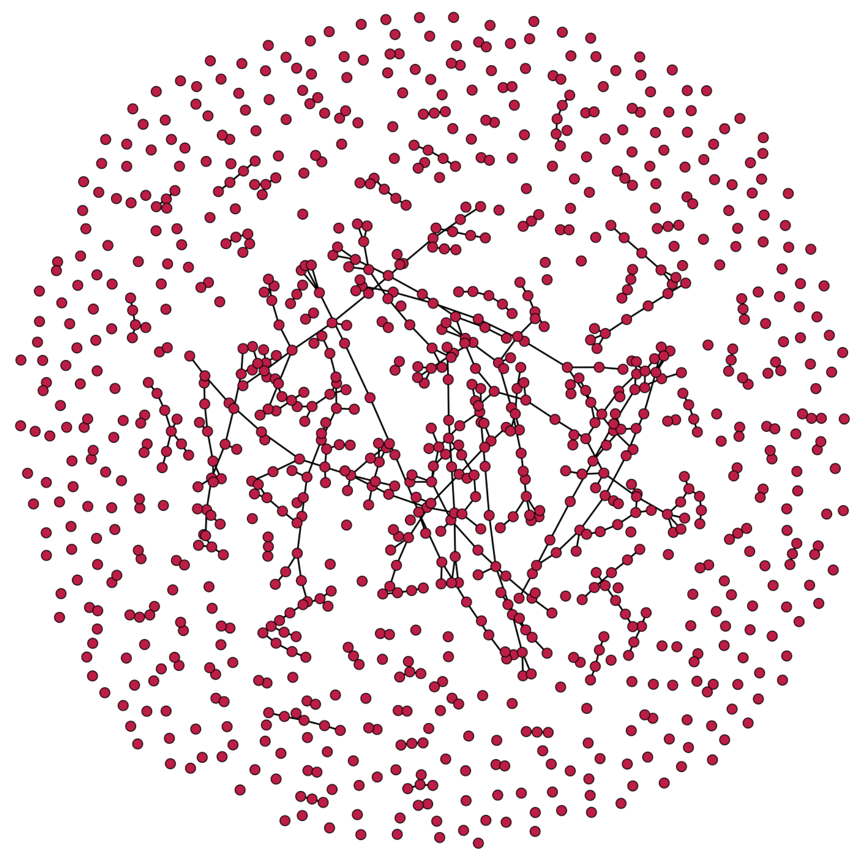
Erdös-Rényi graphs $G(n, p)$

(1959)

- n vertices
- each edge sampled independently w/ probability p

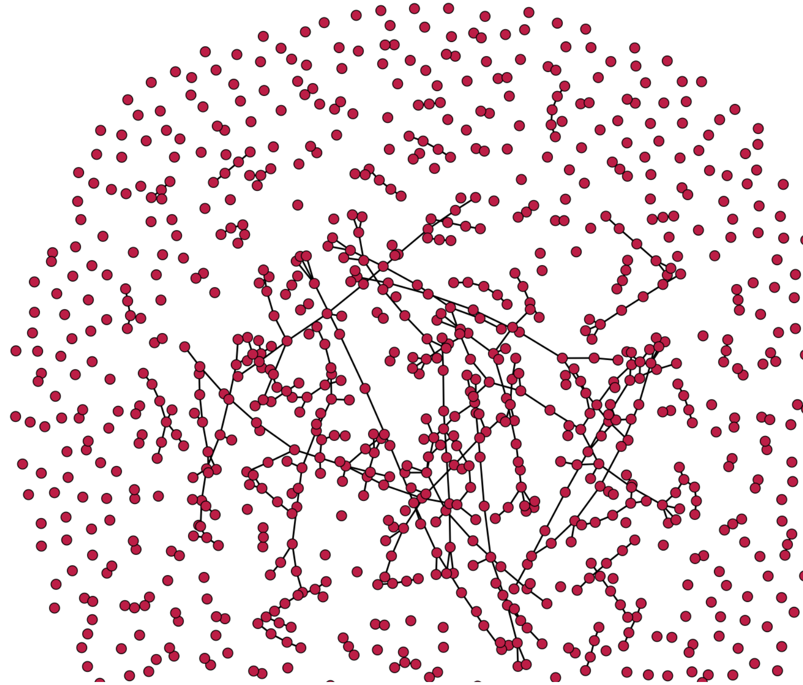
Q: what does such a graph look like?

does it consistently have certain properties?



Erdös-Rényi graphs $G(n, p)$

- n vertices
- each edge sampled independently w/ probability p



Theorem 5.3. *Consider a random graph $G \sim \mathcal{G}(n,p)$ for $p = c/n$, where c is a constant.*

1. *If $c > 1$, then with high probability, there is exactly one connected component of G with $\Omega(n)$ vertices, and all other components have size $\leq O(\log n)$.*

2. *For $c < 1$, then with high probability, all connected components of G have size $< O(\log n)$.*

"threshold phenomena"

Theorem 5.3. *Consider a random graph $G \sim \mathcal{G}(n,p)$ for $p = c/n$, where c is a constant.*

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Proof outline ($c>1$)

① Gap theorem let $p = (1+\varepsilon)/n$. w/ high probability, all components have size

"small component" $\leq O\left(\frac{\log n}{\varepsilon^2}\right)$ or "large component" $\geq \Omega(\varepsilon n)$

② Existence of large component

③ Uniqueness of large component

① Gap theorem let $p=(1+\varepsilon)/n$. w/ high probability, all components have size

$$\leq O\left(\frac{\log n}{\varepsilon^2}\right) \quad \text{or} \quad \geq \Omega(\varepsilon n)$$

Setup

fix vertex v.

let $C(v) \subseteq V$ = connected component of v

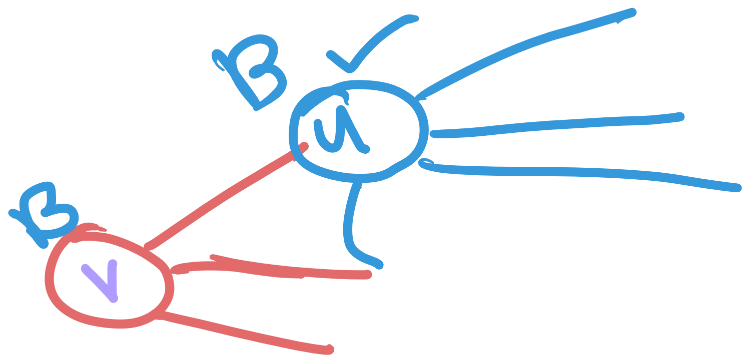
to analyze $C(v)$, imagine revealing $C(v)$ w/ a search algo fix vertex v. $C(v) \subseteq V$ = connected component of v

"Search algo"

$A = \{\text{vertices known to be in } C(v)\}$ (initially $A = \{v\}$)

$B = \{\text{explored vertices}\}$ (initially $B = \emptyset$)

1. while there is a vertex $v_i \in A \setminus B$
   - a. add all neighbors of v_i to A
   - b. add v_i to B



Indexed version of "Search algo"

$A_i = \{\text{vertices known to be in } C(v) \text{ after } i \text{ iter.}\}$ $(A_0 = \{v\})$

$B_i = \{\text{explored vertices after } i \text{ iterations}\}$ $(B_0 = \phi)$

1. For $i = 1, 2, \ldots$ until $A_i = B_i$
   a. Let $v_i \in A_{i-1} \setminus B_{i-1}$
   b. $A_i = A_{i-1} \cup N(v_i)$
   c. $B_i = B_{i-1} \cup \{v_i\}$

A_i = {vertices known to be in $C(v)$ after i iter.} ($A_0 = \{v\}$)

B_i = {explored vertices after i iterations} ($B_0 = \emptyset$)

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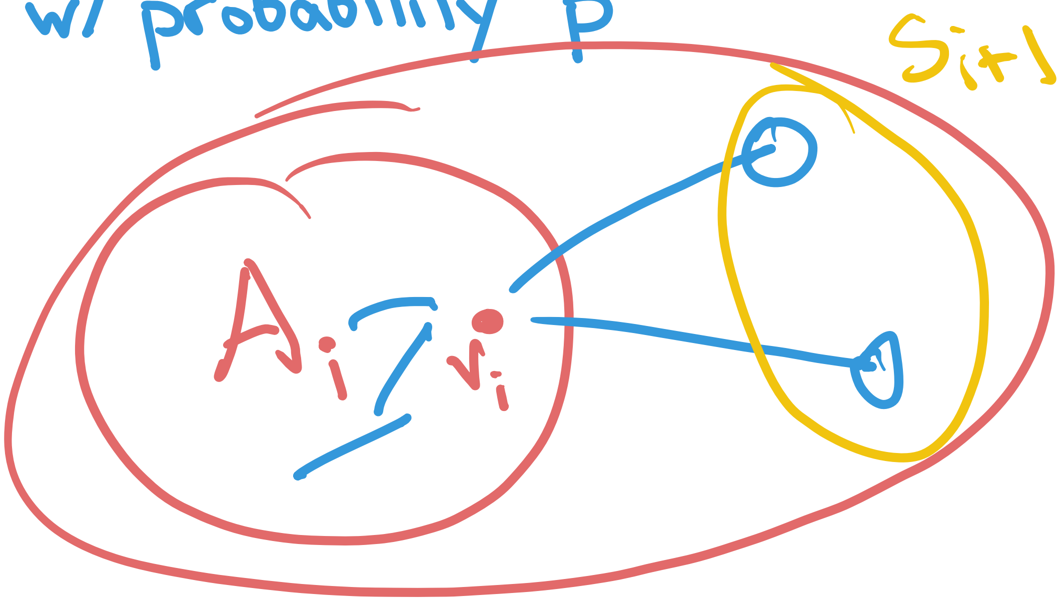
Observation 1

algo terminates iff $|A_i| = i$

Observation 2

$A_{i+1} = A_i \cup S_{i+1}$

where S_{i+1} samples each $v \in V \setminus A_i$ independently w/ probability p



Observation 1 algo terminates iff $|A_i| = i$

Observation 2 $A_{i+1} = A_i \cup S_{i+1}$

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   c. $B_i = B_{i-1} \cup \{v_i\}$

1. For $i = 1, 2, \ldots$ ~~until $|A_i| = i$~~
   a. Let S_i randomly sample $V \setminus A_{i-1}$ w/ probability p
   b. $A_i = A_{i-1} \cup S_i$

$$p_i = \frac{(1+\varepsilon)}{\cancel{n}} \cdot \frac{\varepsilon \cancel{n}}{4} \leq \frac{\varepsilon + \varepsilon^2}{4}$$ (equivalent)

let $i \leq \frac{\varepsilon n}{4}$. $p = \frac{1+\varepsilon}{n}$, $\varepsilon > 0$

① Gap theorem let $\boxed{p = (1+\varepsilon)/n}$ w/ high probability, all components have size $\leq O(\frac{\log n}{\varepsilon^2})$ or $\geq \Omega(\varepsilon n)$

Multiplicative Chernoff bound

let $X_1, \dots, X_n \in [0,1]$ be independent

$\mu = E[X_1 + \dots + X_n]$ $\varepsilon \in (0,1)$

$P[X_1 + \dots + X_n \leq (1-\varepsilon)\mu] \leq e^{-\varepsilon^2 \mu / 2}$

1. For $i = 1, 2, \dots$
   a. Let S_i randomly sample $V \setminus A_{i-1}$ w/ probability p
   b. $A_i = A_{i-1} \cup S_i$

Lemma 1. $E[|A_i|] \geq (1+\frac{\varepsilon}{4})i$

$|A_0| = 1$

$E[|A_1|] = 1 + (n-1)p$

$E[|A_2|] =$

$1 + (1-(1-p)^2)(n-1)$

$E[|A_i|] = 1 + (1-(1-p)^i)(n-1) \geq (1+\frac{\varepsilon}{4})i$

$(1-p)^i \leq e^{-pi} \leq 1 - pi + (pi)^2$ $\leq 1 + x + x^2$ $\forall x \leq \ln(2)$

$\leq 1 - pi + \frac{(\varepsilon+\varepsilon^2)}{4} pi$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

let $i \le \frac{\varepsilon n}{4}$. $\le 1 - \frac{\varepsilon}{4} p_i$

① Gap theorem let $p = (1+\varepsilon)/n$. w/ high probability, all components have size $\le O(\frac{\log n}{\varepsilon^2})$ or $\ge \Omega(\varepsilon n)$

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1. For $i = 1, 2, \dots$ ~~until $|A_i| \ge$~~
   a. Let S_i randomly sample $V \setminus A_{i-1}$ w/ probability p
   b. $A_i = A_{i-1} \cup S_i$

Lemma 1. $E[|A_i|] \ge (1 + \frac{\varepsilon}{4}) i$

Lemma 2.

$$P[|A_i| \le i] \le e^{-\varepsilon^2 i / 8}$$

$$X_j = \begin{cases} 1 & \text{if } j\text{th vertex in } A_i \\ 0 & \text{otherwise} \end{cases}$$

$|A_i| = \sum X_j$, $X_j \in [0,1]$, indep.

$$\frac{1}{1-x} \ge 1 + \frac{x}{2}$$

$$\Pr\left[|A_i| \le \frac{1}{(1+\frac{\varepsilon}{4})} E[|A_i|]\right] \le \Pr\left[|A_i| \le (1 - \tfrac{\varepsilon}{8})\mu\right]$$

$$e^{-\frac{\varepsilon^2}{64 \cdot 2} \cdot \mu} = e^{-(\varepsilon^2/128)\mu}$$

let $i \le \frac{\varepsilon n}{4}$.

Lemma 1. $E[|A_i|] \ge (1+\varepsilon)i$

Lemma 2.

$$P[|A_i| \le i] \le e^{-\varepsilon^2 i/8}$$

for $i \ge 32\log n/\varepsilon^2$, $\le \frac{1}{n^4}$

by union bound over $\frac{32\log n}{\varepsilon^2} \le i \le \frac{\varepsilon n}{4}$,

$$P\left[\frac{32\log n}{\varepsilon^2} \le |A_i| \le \frac{\varepsilon n}{4}\right] \le \frac{1}{n^3}$$

① Gap theorem let $p=(1+\varepsilon)/n$. w/ high probability, all components have size $\le O(\frac{\log n}{\varepsilon^2})$ or $\ge \Omega(\varepsilon n)$

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let $X_1, \dots, X_n \in [0,1]$ be independent

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Proof outline ($c>1$)

① Gap theorem let $p=(1+\varepsilon)/n$. w/ high probability, all components have size

$$\leq O\left(\frac{\log n}{\varepsilon^2}\right) \quad \text{or} \quad \geq \Omega(\varepsilon n)$$

② Existence of large component

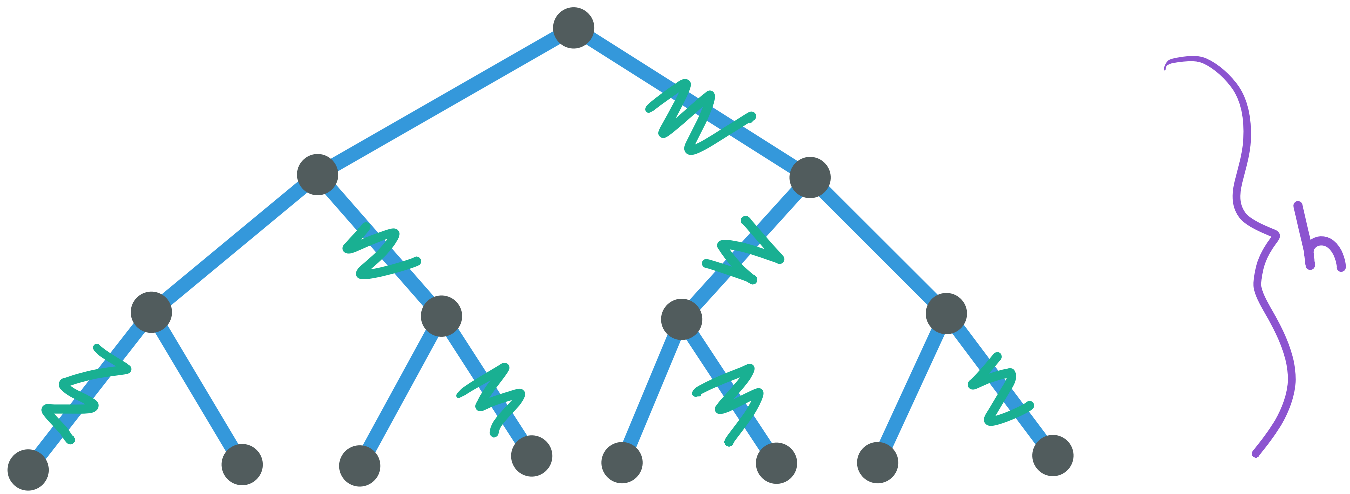
③ Uniqueness of large component

② Existence of large component

Lemma 5.5. *Let $p = (1+\epsilon)/n$ for $\epsilon > 0$. Let $v \in V$ be a vertex. For all $3 \leq h \leq \epsilon n$, with probability at least $1/h$, v has at least $1 + h$ vertices in its connected component.*

Lemma 5.5. *Let $p = (1+\epsilon)/n$ for $\epsilon > 0$. Let $v \in V$ be a vertex. For all $3 \le h \le \epsilon n$, with probability at least $1/h$, v has at least $1+h$ vertices in its connected component.*

Last time



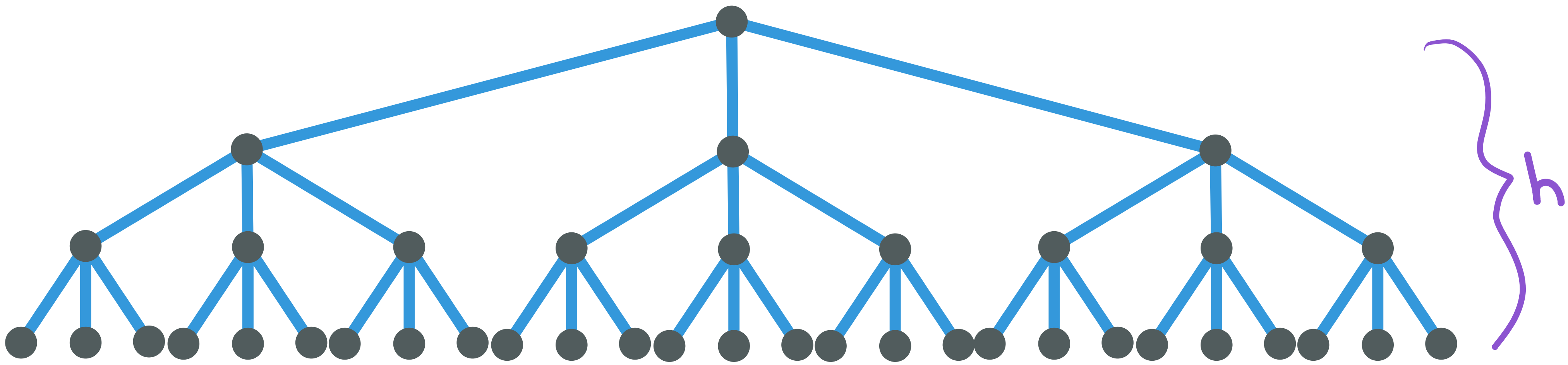
let T = complete binary tree of height h,

w/ every edge deleted w/ prob $\le 1/2$

Fact: $P[\text{some leaf connected to root}] \ge \frac{1}{h}$

"Galton-Watson process"

Lemma 5.5. *Let $p = (1+\epsilon)/n$ for $\epsilon > 0$. Let $v \in V$ be a vertex. For all $3 \le h \le \epsilon n$, with probability at least $1/h$, v has at least $1 + h$ vertices in its connected component.*

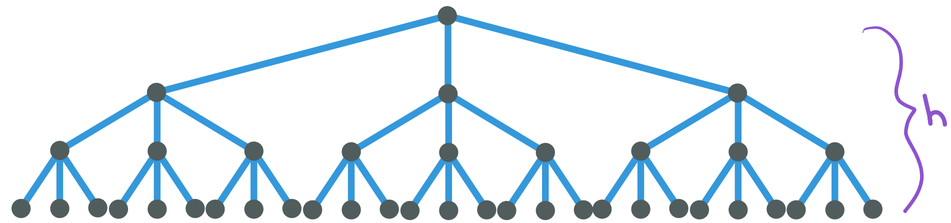


let T = complete k-ary tree of height h, w/ every edge deleted w/ prob $\le 1/k$

Fact: $P[\text{some leaf connected to root}] \ge \frac{1}{h}$

"Galton-Watson process"

Lemma 5.5. *Let $p = (1+\epsilon)/n$ for $\epsilon > 0$. Let $v \in V$ be a vertex. For all $3 \le h \le \epsilon n$, with probability at least $1/h$, v has at least $1+h$ vertices in its connected component.*



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w/ every edge deleted w/ prob $\le 1/k$

Fact: $P[\text{some leaf connected to root}] \ge \frac{1}{h}$

"Galton-Watson process"

Equivalent process

start w/ population of size 1

Each generation, each individual flips k coins w/ prob. $1/k$ of heads

total # heads gives size of next generation

Fact: $P[\text{surviving } h \text{ generations}] \ge \frac{1}{h}$

Lemma 5.5. *Let $p = (1+\epsilon)/n$ for $\epsilon > 0$. Let $v \in V$ be a vertex. For all $3 \le h \le \epsilon n$, with probability at least $1/h$, v has at least $1 + h$ vertices in its connected component.*

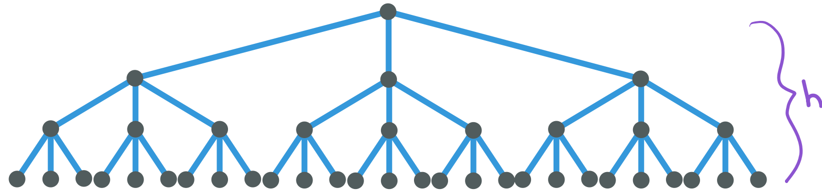
Recall search algo.

conditional on $|A_{i-1}| \le h$,

A_i / A_{i-1} is like sampling

$n - h \ge \frac{n}{1+\varepsilon}$ children each

w/ prob. $p = \frac{1+\varepsilon}{n}$



let T = complete k-ary tree of height h,
w/ every edge deleted w/ prob $\le 1/k$

Fact: $P[\text{some leaf connected to root}] \ge \frac{1}{h}$

"Galton-Watson process"

1. For $i = 1, 2, \dots$ until $A_i = B_i$
   a. Let $v_i \in A_{i-1} \setminus B_{i-1}$
   b. $A_i = A_{i-1} \cup N(v_i)$
   c. $B_i = B_{i-1} \cup \{v_i\}$

Theorem 5.3. *Consider a random graph $G \sim \mathcal{G}(n,p)$ for $p = c/n$, where c is a constant.*

1. *If $c > 1$, then with high probability, there is exactly one connected component of G with $\Omega(n)$ vertices, and all other components have size $\leq O(\log n)$.*
2. *For $c < 1$, then with high probability, all connected components of G has size $< O(\log n)$.*

Proof outline ($c>1$)

① Gap theorem let $p=(1+\varepsilon)/n$. w/ high probability, all components have size

$$\leq O\left(\frac{\log n}{\varepsilon^2}\right) \quad \text{or} \quad \geq \Omega(\varepsilon n)$$

② Existence of large component

③ Uniqueness of large component

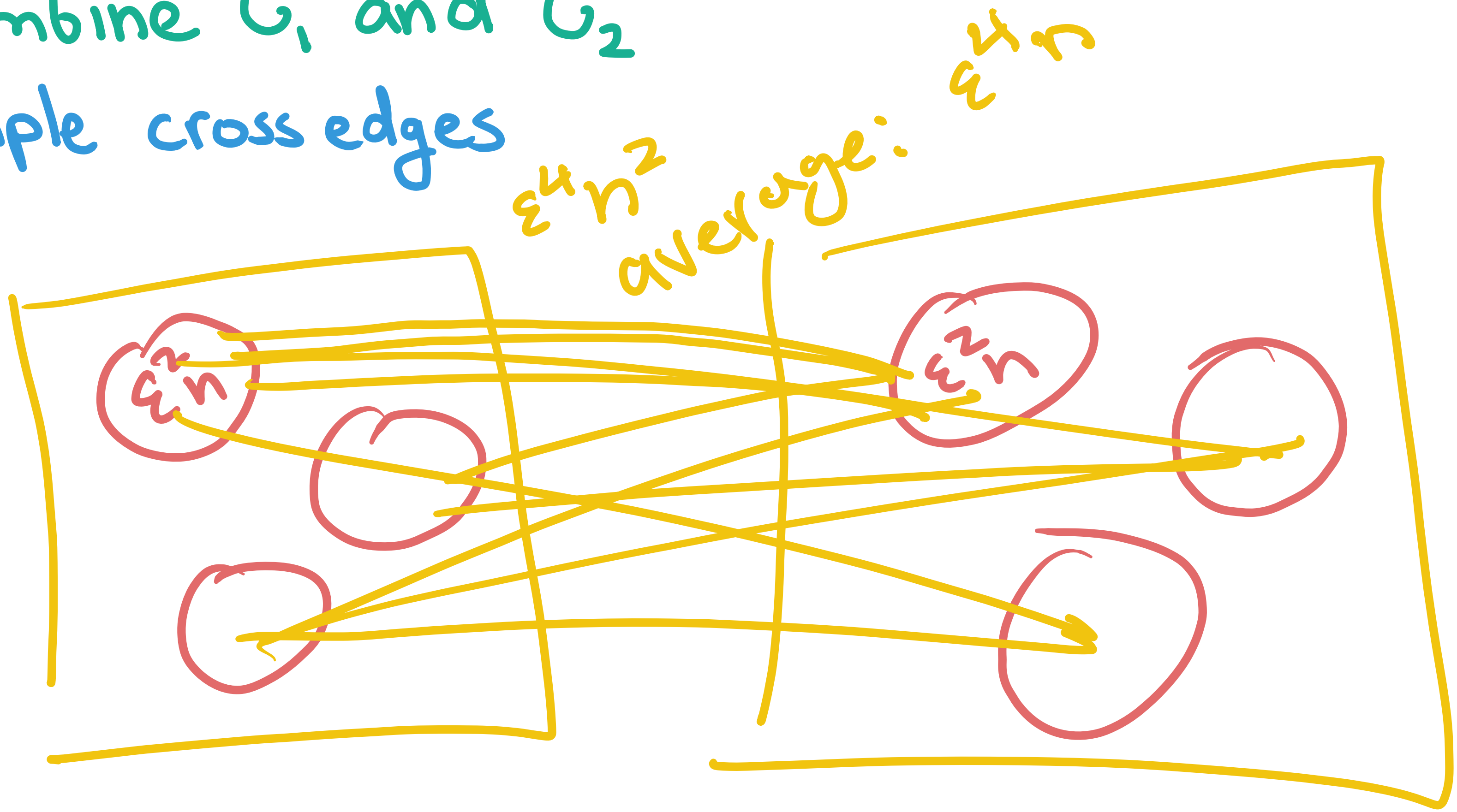
Imagine sampling $G \sim (n,p)$ as:

① sample $G_1 \sim (n/2, p)$

② sample $G_2 \sim (n/2, p)$

③ combine G_1 and G_2

④ sample cross edges



Multiplicative Chernoff bound

let $X_1, \dots, X_n \in [0,1]$ be independent

$\mu = E[X_1 + \dots + X_n]$

$\varepsilon \in (0,1)$

- $P[X_1 + \dots + X_n \geq (1+\varepsilon)\mu] \leq e^{-\varepsilon^2 \mu/3}$

- $P[X_1 + \dots + X_n \leq (1-\varepsilon)\mu] \leq e^{-\varepsilon^2 \mu/2}$

(proof at end if time permits)

models, these graphs – however random – tend to be extremely consistent about certain properties.

Today we will study the $\mathcal{G}(n,p)$ random graph, sometimes called Erdös-Rényi graphs based on work by Erdös and Rényi [ER59; ER60]. A random graph from $\mathcal{G}(n,p)$ is an undirected graph over n vertices, where every edge is sampled independently with probability p. By now there is a large catalog of nontrivial and useful properties that, depending on p, are almost certain to appear or not appear in such a graph (for sufficiently large n). Moreover, Erdös and Rényi showed that these properties can vary dramatically with very small changes in p. Consider the following theorem.

Theorem 2.1. *Consider a random graph $G \sim \mathcal{G}(n,p)$ for $p = c/n$, where c is a constant.*

1. *If $c > 1$, then with high probability, there is exactly one connected component of G with $\Omega(n)$ vertices, and all other components have size $\leq O(\log n)$.*
2. *For $c < 1$, then with high probability, all connected components of G have size $< O(\log n)$.*

The parameter c above models the average degree (in expectation). The drama lies in the fact that a tiny change in the average degree c – from .999 to 1.0001 – flips the qualitative nature of a typical random graph from one of many tiny components to essentially one giant component. This is an example of a *threshold phenomena*; alternatively, a *nonlinear dynamic*. Such phenomena is not rare: it occurs in many situations in physics, as well as in models for epidimiology and social networks. Let us briefly mention — without claiming to be very precise — that the sensitivity to c gives some motivation for controlling the "reproductive number" when analyzing and preventing the spread of infectious diseases. The reproductive number is the expected number of healthy individuals that a sick individual effects.

We note that further research has obtained a much more refined and detailed understanding than stated in theorem 2.1. We refer the reader to [Bol98, Chapter 7] for further details and other results in this area.

We present the proof for $c > 1$. The second case of $c < 1$ is left to the reader as exercise 1.

2.1 Overview of the proof for $c > 1$

We will prove part 1 of theorem 2.1 in roughly three parts.

Part 1: the gap theorem. Observe that in theorem 2.1 above, regardless of the value of p, there are simply no "medium"-size components, such as a component of size $\sqrt{n}$ or of size $n/\log(n)$. The intermediate sizes are ruled out by the following "gap theorem".

Lemma 2.2. *There is a universal constant $C > 0$, such that for all $\epsilon \in (0,1)$, and for all $n > 0$ sufficiently large, and $p = (1+\epsilon)/n$, we have the following. For a random graph* designed elebarate randomized constructions of graphs and showed that with nonzero probability, they can possess certain counterintuitive, seemingly impossible properties. This general approach is now called *Ramsey theory*. Second, he showed that for natural random graph models, these graphs – however random – tend to be extremely consistent about certain properties.

Today we will study the $\mathcal{G}(n, p)$ random graph, sometimes called Erdös-Rényi graphs based on work by Erdös and Rényi [ER59; ER60]. A random graph from $\mathcal{G}(n, p)$ is an undirected graph over n vertices, where every edge is sampled independently with probability p. By now there is a large catalog of nontrivial and useful properties that, depending on p, are almost certain to appear or not appear in such a graph (for sufficiently large n). Moreover, Erdös and Rényi showed that these properties can vary dramatically with very small changes in p. Consider the following theorem.

Theorem 5.3. *Consider a random graph $G \sim \mathcal{G}(n, p)$ for $p = c/n$, where c is a constant.*

1. *If $c > 1$, then with high probability, there is exactly one connected component of G with $\Omega(n)$ vertices, and all other components have size $\leq O(\log n)$.*
2. *For $c < 1$, then with high probability, all connected components of G have size $< O(\log n)$.*

The parameter c above models the average degree (in expectation). The drama lies in the fact that a tiny change in the average degree c – from .999 to 1.0001 – flips the qualitative nature of a typical random graph from one of many tiny components to essentially one giant component. This is an example of a *threshold phenomena*; alternatively, a *nonlinear dynamic*. Such phenomena is not rare: it occurs in many situations in physics, as well as in models for epidimiology and social networks. Let us briefly mention — without claiming to be very precise — that the sensitivity to c gives some motivation for controlling the "reproductive number" when analyzing and preventing the spread of infectious diseases. The reproductive number is the expected number of healthy individuals that a sick individual effects.

We note that further research has obtained a much more refined and detailed understanding than stated in theorem 5.3. We refer the reader to [Bol98, Chapter 7] for further details and other results in this area.

We present the proof for $c > 1$. The second case of $c < 1$ is left to the reader as exercise 5.1.

5.2.1 Overview of the proof for $c > 1$

We will prove part 1 of theorem 5.3 in roughly three parts.